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Eigenvalues for the Equation of Species Conservation with Heterogeneous Reaction

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Fox and Libby[1] consider several problems involving the energy equation for laminar boundary layers with velocity fields described by the Blasius function $f_0(\eta)$; in particular they present solutions, which are in principle exact, to the following two initial value problems. Consider the stagnation enthalpy distribution $g(s, \eta)$, $g \equiv h_s/h_{s,e}$, in a boundary layer wherein the $\rho\mu$ ratio is constant ($C \simeq 1$) and the Prandtl number is unity, and where, at an initial station $s = s_0 > 0$, an arbitrary distribution of stagnation enthalpy is given, i.e., $g(s_0, \eta)$ is specified. The two different problems arise because there are considered two boundary conditions at the wall in the downstream region $s > s_0$. In one, the wall enthalpy in terms of g_w is constant, and in the second, the heat transfer [more precisely $(\partial g/\partial \eta)_w$] is zero. The solution to each of these problems is given in terms of a complete set of orthogonal functions, the first ten of which in each set are presented in Ref. 1.

It is also indicated there that the solutions for the energy distribution just outlined can be applied to certain related problems involving species and/or element conservation. For this application it is necessary to make the additional assumption that a single diffusion coefficient exists with a Schmidt number based thereon equal to unity. The two boundary conditions correspond, respectively, to a fully catalytic surface and to a noncatalytic surface. The question of whether conservation of species or of elements is described depends, of course, on whether or not gas phase reaction exists.

It is the purpose of this note to outline an extension of these considerations to the case of boundary layers with no gas phase reaction but with finite surface catalyticity in the downstream region $s > s_0$ and to present a number of eigenvalues that pertain thereto. As will be discussed below, to obtain the related eigenfunctions involves little numerical effort; thus, the presentation of the eigenvalues, whose determination is not so simple, is considered to be of some value.

For orientation, consider the following mathematical problem:

$$\frac{\partial^3 Y_i}{\partial \eta^3} + f_0 \frac{\partial Y_i}{\partial \eta} - 2sf_0' \frac{\partial Y_i}{\partial s} = 0 \qquad (1)$$

$$Y_i(s_0, \eta) = Y_{i,0}(\eta), \text{ given arbitrary function}$$

$$Y_i(s, \infty) = Y_{i,e}, \text{ given const}$$

$$\frac{\partial Y_i}{\partial \eta}(s, 0) = \zeta_i Y_i(s, 0) \qquad \zeta_i, \text{ given const}$$

where Y_i is the mass fraction of species i, s and η are the usual Levy-Lees variables, and ζ_i the surface catalyticity. Solution of this mathematical problem describes the concentration distribution of species i in a laminar boundary layer which has velocities given by the Blasius function, flows over a surface with a constant surface catalyticity, and which has an arbitrary initial profile of concentration. Such a profile can be created, for example, by an upstream value of surface catalyticity different from ζ_i. Related problems have been under widespread attack; Ref. 2 is especially relevant to the present note, whereas Ref. 3 provides a current review of flows with heterogeneous reactions. Reference 4 provides, inter alia, a review of recent developments in the theory of laminar boundary layers involving gas phase and surface reactions. The reader is referred to this literature for a discussion of the physical aspects of the flow described here.

A solution to the foregoing problem may be obtained by separation of variables. In the interest of brevity the solution will here be simply written down; however, it is easy to confirm that the solution is

$$Y_i = Y_{i,e}(f_{0,w}'' + \zeta_i f_0')(f_{0,w}'' + \zeta_i)^{-1} + \sum_{n=1}^{\infty} A_{n,i} \left(\frac{s}{s_0}\right)^{-\lambda_{n,i}} N_{n,i}(\eta) \qquad (2)$$

where the $A_{n,i}$ coefficients are determined as shown below from the initial conditions, i.e., at $s = s_0$, and where $\lambda_{n,i}$ and $N_{n,i}$ are the eigenvalues and eigenfunctions, respectively, and are defined by the equation

$$N_{n,i}'' + f_0 N_{n,i}' + \lambda_{n,i} f_0' N_{n,i} = 0 \qquad (3)$$

subject to the conditions

$$N'_{n,i}(0) = \zeta_i N_{n,i}(0) \qquad N_{n,i}(\infty) = 0 \qquad (4)$$

The functions $N_{n,i}(\eta)$ defined by Eqs. (3) and (4) are generalizations of those previously presented in Ref. 1, since for $\zeta_i \rightarrow \infty$ there result the eigenfunctions for the case of constant wall enthalpy and for $\zeta_i = 0$ the eigenfunctions for the case of an adiabatic surface.

As demonstrated in Ref. 1, well-known procedures can be used to show that, if exponential decay to zero of the eigenfunctions is required as $\eta \rightarrow \infty$, then these functions form a complete orthogonal set and the eigenvalues are real positive numbers. Thus, as $s/s_0 \rightarrow \infty$ in Eq. (2), the solution becomes that corresponding to a Crocco relation between concentration and velocity. Moreover, the orthogonality condition is found to be

$$\int_0^\infty \left(\frac{f_0'}{f_0''}\right) N_{n,i} N_{m,i} d\eta = C_{n,i}\delta_{mn} \qquad (5)$$

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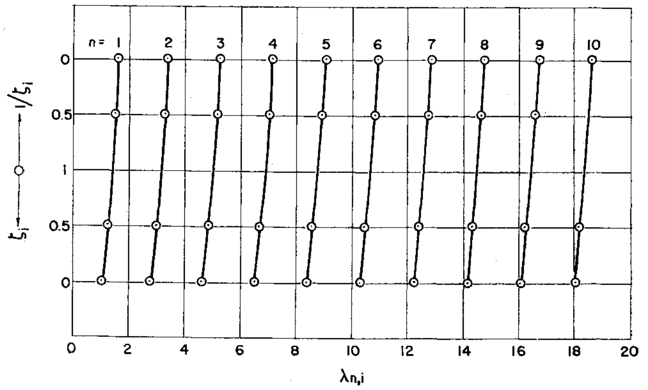

Fig. 1 Variation of eigenvalues with ζ_i.

Equation (5) may be used to evaluate the $A_{n,i}$ coefficients in the manner customary with orthogonal functions. The only restriction on $Y_{i,0}(\eta)$ is that it must decay exponentially at least as fast as $[(\eta - \varkappa)^2/4]$ as $\eta \rightarrow \infty$.

In principle, the solution given by Eqs. (2–4) is exact so that it is possible to compute, for example, the distribution of species concentration downstream of a discontinuity in ζ_i. This problem for the limiting case of infinite Schmidt number has been solved in Ref. 2. Here, of course, the Schmidt number is unity. If the perturbation point of view of Refs. 1 and 5 is followed, there can be computed, with little effort, first- and higher-order corrections to the present solution for deviations of the velocity field from that described by the Blasius function, for variations in the $\rho\mu$ product, and for nonunity Schmidt number, provided that the surface catalyticity is indeed constant.

The most convenient and accurate method of determining the eigenvalues and eigenfunctions defined by Eqs. (3) and (4) has been found to involve use of the asymptotic solution [which applies to Eq. (3)], i.e., the solution when $f_0 \simeq \eta - \varkappa$, $f_0' \simeq 1$, and in which the function with power law decay in the form $(\eta - \varkappa)^{-\lambda_{n,i}}$ is suppressed. By picking arbitrarily a value of $N_{n,i}$ at $\eta = \eta^*$, where η^* satisfies the inequality $|1 - \lambda_{n,i}|(\eta^* - \varkappa)^{-2} \ll 1$, and a value of $\lambda_{n,i}$, the value of $N_n{}',{}_i(\eta^*)$ on the asymptotic solution for $\eta > \eta^*$ may be computed, and the integration in the direction of decreasing η can be carried out. When $\eta = 0$ is reached, the value of $\zeta_i = N_{n,i}'(0)/N_{n,i}(0)$ is obtained. If a particular value of ζ_i is desired, a new guess for $\lambda_{n,i}$ must be made and the integration repeated. On the contrary, if an eigenvalue for a particular value of ζ_i is available, then the integration can proceed from the wall ($\eta = 0$) outward with either $N_{n,i}(0) = 1$ or $N'_{n,i}(0) = 1$, for example, and the eigenfunction and value of the normalizing parameter $C_{n,i}$ obtained from a single integration and quadrature. Clearly, a small scale computer is adequate for this work.

The first ten eigenvalues for $\zeta_i = 0, \infty$ given in Ref. 1 have been supplemented in the present work by the corresponding first ten values for $\zeta_i = \frac{1}{2}$ and the first nine values of $\zeta_i = 4$; these results are presented numerically in Table 1 and graphically in Fig. 1. The significant figures in Table 1 reflect the accuracy with which the specified values of ζ_i have been achieved. The results presented here permit ready estimates to be made for the eigenvalues for any value of ζ_i and for values of $n > 10$; this is suggested by the "smoothness" of the curves of $\lambda_{n,i}$ vs ζ_i for a given n and by the "almost equal" spacing of successive $\lambda_{n,i}$'s, namely, a spacing of 1.94.

Table 1 Eigenvalues for various surface catalyticity

$n \diagdown \zeta_i$	0	$\frac{1}{2}$	4	∞
1	1	1.26	1.50	1.573
2	2.77	3.00	3.29	3.385
3	4.62	4.83	5.14	5.25
4	6.51	6.70	7.02	7.14
5	8.41	8.59	8.92	9.05
6	10.32	10.50	10.82	10.96
7	12.24	12.41	12.74	12.88
8	14.17	14.33	14.66	14.81
9	16.10	16.26	16.59	16.74
10	18.04	18.19	...	18.68

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Reduction of Torsional Stiffness Due to Thermal Stress in Thin, Solid Wings

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RESULTS are given in Ref. 1 which show how the spanwise thermal stresses caused by aerodynamic heating can produce a reduction in the torsional stiffness of thin wing sections. In Ref. 2 the analyses have been extended to include the effects of cross-sectional shape, solidity, and other parameters. The purpose of the present note is to examine the implications of using assumed temperature distributions in such analyses and to determine whether these distributions can be readily correlated with the temperatures that would result at only three points on the wing semichord for an actual aerodynamic heating problem. The notation used and assumptions made are similar to those of Ref. 1.

Consider the symmetric solid section (Fig. 1) whose chordwise thickness variation is given by

$$t = t_0 + \Delta t|\beta|^m \tag{1}$$

and let the chordwise temperature rise distribution be approximated by

$$T = T_0 + \Delta T|\beta|^n \tag{2}$$

The spanwise thermal stresses are determined from the simple equation

$$\sigma_{yy} = E\alpha[\textstyle\int_A T dA/A - T] \tag{3}$$

and the effective torsional stiffness from the equation

$$GJ_{\text{eff}}/GJ_0 = 1 + [\textstyle\int_A \sigma_{yy} r^2 dA/GJ_0] \tag{4}$$

If Eqs. (1) and (2) are substituted into Eq. (3) and then into Eq. (4), with $r \simeq x$ and $dA = t\,dx$, one obtains

$$\frac{GJ_{\text{eff}}}{GJ_0} = 1 - \left(\frac{E}{G}\right)\left(\frac{c}{t_0}\right)^2\left(\frac{\alpha\Delta T}{4}\right)K \tag{5}$$

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